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On some integrable systems related to the Toda lattice

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Abstract. We discuss some of the integrable lattices introduced recently by R Yamilov. We demonstrate that they are closely related to the usual Toda lattice by means of a sort of Bäcklund transformations. We also apply the general procedure of integrable discretization and obtain their integrable finite-difference approximations. These novel integrable discrete-time systems are also related to the discrete-time Toda lattice by means of the Bäcklund transformations. The whole construction exploits the tri-Hamiltonian structure of the Toda lattice.

1. Introduction

In a recent paper, Yamilov [1] gave a complete list of lattice systems of the form

$$\ddot{x}_k = f(\dot{x}_k, x_{k+1}, x_k, x_{k-1})$$

having infinitely many local conservation laws (in fact, he demonstrated that the existence of only *two* conservation laws of high enough orders leads to the same list of lattices). Up to simple transformations the list consists of only two items, one of them being the family of lattices

$$\ddot{x}_k = R(\dot{x}_k)[g(x_{k+1} - x_k) - g(x_k - x_{k-1})] \tag{1.1}$$

where $R(u) = \epsilon u^2 + \alpha u + \beta$ and the function $g(x)$ satisfies the ordinary differential equation $g' = \epsilon g^2 + \gamma g + \delta$.

Clearly this family generalizes the most celebrated and well-studied integrable lattice system, the Toda lattice:

$$\ddot{x}_k = \exp(x_{k+1} - x_k) - \exp(x_k - x_{k-1}) \tag{1.2}$$

which corresponds to $R(u) = 1$, $g(x) = \exp(x)$, $g' = g$.

We will consider two more complicated representatives of the family (1.1) (which exhaust a large part of the whole family upon linear changes of the independent and dependent variables). Namely, we consider two cases: first, $R(u) = u$, $g(x) = \exp(x)$, $g' = g$ and second, $R(u) = -u^2$, $g(x) = \coth(x)$, $g' = -g^2 + 1$. In other words, we consider the lattices

$$\ddot{x}_k = \dot{x}_k(\exp(x_{k+1} - x_k) - \exp(x_k - x_{k-1})) \tag{1.3}$$

$$\ddot{x}_k = -\dot{x}_k^2(\coth(x_{k+1} - x_k) - \coth(x_k - x_{k-1})). \tag{1.4}$$

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(Note under the rescaling $x \mapsto \nu x$ and subsequent sending $\nu \to 0$ the last system turns into

$$\ddot{x}_k = -\dot{x}_k^2 \left(\frac{1}{x_{k+1} - x_k} - \frac{1}{x_k - x_{k-1}} \right)$$

also belonging to the family (1.1) with $R(u) = -u^2$, $g(x) = 1/x$, $g' = -g^2$. We will not consider this system separately, always having in mind that it is a limiting case of (1.4).)

Our aim here is to point out an unexpected and surprising circumstance; both lattices (1.3) and (1.4) turn out to be closely related to the usual Toda lattice. More precisely we will demonstrate that three lattices (1.2), (1.3) and (1.4) are three different appearances of one and the same tri-Hamiltonian system. They correspond to three different parametrizations of the phase variables of this system by means of canonically conjugated coordinates and momenta, resulting in three different (compatible) invariant Poisson structures.

Moreover, this observation may be prolonged to the level of integrable discretizations. In the following difference equations $x_k = x_k(t)$ are supposed to be functions of the discrete time $t \in h\mathbb{Z}$, and $\tilde{x}_k = x_k(t+h)$, $\underset{\sim}{x}_k = x_k(t-h)$. In [2] the author introduced a method of constructing integrable discretizations of the systems allowing an r-matrix interpretation, and applied this method to the Toda lattice (1.2). An outcome consisted of two different discretizations,

$$\exp(\tilde{x}_k - x_k) - \exp(x_k - \underset{\sim}{x}_k) = h^2(\exp(\underset{\sim}{x}_{k+1} - x_k) - \exp(x_k - \tilde{x}_{k-1})) \quad (1.5)$$

and

$$\exp(\tilde{x}_k - 2x_k + \underset{\sim}{x}_k) = \frac{1 + h^2 \exp(\underset{\sim}{x}_{k+1} - x_k)}{1 + h^2 \exp(x_k - \tilde{x}_{k-1})}. \quad (1.6)$$

We will demonstrate that exploiting the tri-Hamiltonian structure of the corresponding discrete-time system, one can also obtain discretizations for (1.3) and (1.4)

$$\frac{\exp(\tilde{x}_k - x_k) - 1}{\exp(x_k - \underset{\sim}{x}_k) - 1} = \frac{1 + h \exp(\underset{\sim}{x}_{k+1} - x_k)}{1 + h \exp(x_k - \tilde{x}_{k-1})} \quad (1.7)$$

and

$$\coth(\tilde{x}_k - x_k) - \coth(x_k - \underset{\sim}{x}_k) = \coth(\underset{\sim}{x}_{k+1} - x_k) - \coth(x_k - \tilde{x}_{k-1}) \quad (1.8)$$

respectively.

Finally, we will consider the explicit integrable discretizations for all three lattices, which arise if one replaces on the right-hand sides of the above difference equations $\underset{\sim}{x}_{k+1}$, $\tilde{x}_{k-1}$ by x_{k+1}, x_{k-1}, respectively. We will demonstrate that such a change leads to a different tri-Hamiltonian discrete-time system, whose natural phase space is that of the *relativistic* Toda lattice, and elaborate the corresponding parametrizations.

All the systems above (continuous and discrete-time ones) may be considered either on an infinite lattice ($k \in \mathbb{Z}$), or on a finite one ($1 \leqslant k \leqslant N$). In the last case one of the two types of boundary conditions may be imposed: open-end ($x_0 = \infty$, $x_{N+1} = -\infty$) or periodic ($x_0 \equiv x_N$, $x_{N+1} \equiv x_1$). We shall only be concerned with the finite lattices here, consideration of the infinite ones being to a large extent similar.

2. Newtonian equations of motion: Lagrangian and Hamiltonian formulations

All the equations introduced in the previous section, both continuous- and discrete-time, are written in the Newtonian form:

$$\ddot{x}_k = \Phi_k(\dot{x}, x) \qquad \text{or} \qquad \Psi_k(\tilde{x}, x, \underset{\sim}{x}) = 0$$

respectively. They all turn out to admit a Lagrangian formulation.

Recall that in the continuous time case Lagrangian equations are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}_k}-\frac{\partial\mathcal{L}}{\partial x_k}=0 \tag{2.1}$$

while their discrete-time analogue is given by

$$\partial(\Lambda(\tilde{x},x)+\Lambda(x,\underset{\sim}{x}))/\partial x_k=0. \tag{2.2}$$

Further, recall that Lagrangian formulation also implies a possibility of introducing a Hamiltonian one. Namely, in the continuous time case one defines momenta p_k canonically conjugated to the coordinates x_k by

$$p_k=\partial\mathcal{L}/\partial\dot{x}_k. \tag{2.3}$$

Then the flow defined by (2.1), being expressed in terms of (x,p), preserves the standard symplectic form $\sum \mathrm{d}x_k\wedge\mathrm{d}p_k$ on the phase space $\mathbb{R}^{2N}(x,p)$. Moreover, this flow may be written in a canonical form

$$\dot{x}_k=\partial H/\partial p_k \qquad \dot{p}_k=-\partial H/\partial x_k \tag{2.4}$$

the Hamiltonian function $H(x,p)$ being given by

$$H=\sum_{k=1}^{N}\dot{x}_k p_k-\mathcal{L}. \tag{2.5}$$

Analogously, in the discrete-time case the momenta p_k canonically conjugated to x_k are given by

$$p_k=\partial\Lambda(x,\underset{\sim}{x})/\partial x_k. \tag{2.6}$$

Then the map $(x,\underset{\sim}{x})\mapsto(\tilde{x},x)$ induces a symplectic map $(x,p)\mapsto(\tilde{x},\tilde{p})$ of the phase space $\mathbb{R}^{2N}(x,p)$, i.e. a map preserving the standard symplectic form $\sum \mathrm{d}x_k\wedge\mathrm{d}p_k$. Note that (2.6) implies that the equations (2.2) may be presented as

$$p_k=-\partial\Lambda(\tilde{x},x)/\partial x_k \tag{2.7}$$

$$\tilde{p}_k=\partial\Lambda(\tilde{x},x)/\partial\tilde{x}_k. \tag{2.8}$$

3. Simplest flow of the Toda hierarchy and its tri-Hamiltonian structure

All three lattices (1.2), (1.3) and (1.4) arise from the simplest flow of the Toda hierarchy under different parametrizations of the relevant variables (a,b) (called Flaschka variables) by the canonically conjugated variables (x,p).

The simplest flow of the Toda hierarchy (hereafter denoted by TL) is

$$\dot{a}_k=a_k(b_{k+1}-b_k) \qquad \dot{b}_k=a_k-a_{k-1}. \tag{3.1}$$

It may be considered either under open-end boundary conditions ($a_0=a_N=0$), or under periodic ones (all the subscripts are taken (mod N), so that $a_0\equiv a_N$, $b_{N+1}\equiv b_1$).

Its discretization introduced in [2, 3] (and called hereafter DTL) is given by the difference equations

$$\tilde{a}_k=a_k\frac{\beta_{k+1}}{\beta_k} \qquad \tilde{b}_k=b_k+h\left(\frac{a_k}{\beta_k}-\frac{a_{k-1}}{\beta_{k-1}}\right) \tag{3.2}$$

where $\beta_k=\beta_k(a,b)$ are defined as a unique set of functions satisfying the recurrent relation

$$\beta_k=1+hb_k-\frac{h^2a_{k-1}}{\beta_{k-1}} \tag{3.3}$$

together with an asymptotic relation

$$\beta_k = 1 + hb_k + \mathrm{O}(h^2). \tag{3.4}$$

In the open-end case, owing to $a_0 = 0$, we obtain from (3.3) the following finite continued fractions expressions for β_k:

$$\beta_1 = 1 + hb_1; \qquad \beta_2 = 1 + hb_2 - \frac{h^2 a_1}{1 + hb_1}; \qquad \ldots$$

$$\beta_N = 1 + hb_N - \cfrac{h^2 a_{N-1}}{1 + hb_{N-1} - \cfrac{h^2 a_{N-2}}{1 + hb_{N-2} - \ddots - \cfrac{h^2 a_1}{1 + hb_1}}}.$$

In the periodic case (3.3) and (3.4) uniquely define β_ks as N-periodic infinite continued fractions. It can be proved that for h small enough these continued fractions converge and their values satisfy (3.4).

Let us recall the Lax representations of the flow TL and of the map DTL. They are given in terms of the $N \times N$ Lax matrix T depending on the phase space coordinates a_k, b_k and (in the periodic case) on the additional parameter λ

$$T(a, b, \lambda) = \sum_{k=1}^{N} b_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} + \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1}. \tag{3.5}$$

Here E_{jk} stands for the matrix whose only non-zero entry on the intersection of the jth row and the kth column is equal to one. In the periodic case we have $E_{N+1,N} = E_{1,N}$, $E_{N,N+1} = E_{N,1}$; in the open-end case we set $\lambda = 1$, and $E_{N+1,N} = E_{N,N+1} = 0$.

The flow (3.1) is equivalent to the matrix differential equation

$$\dot{T} = [T, B] \tag{3.6}$$

where

$$B(a, b, \lambda) = \sum_{k=1}^{N} b_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \tag{3.7}$$

and the map (3.2) is equivalent to the matrix difference equation

$$\tilde{T} = \mathbf{B}^{-1} T \mathbf{B} \tag{3.8}$$

where

$$\mathbf{B}(a, b, \lambda) = \sum_{k=1}^{N} \beta_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k}. \tag{3.9}$$

The spectral invariants of the matrix $T(a, b, \lambda)$ serve as integrals of motion for the flow TL, as well as for the map DTL.

It turns out that the flow TL is Hamiltonian with respect to three different compatible Poisson brackets and the map DTL is also Poisson with respect to them. The spectral invariants of the matrix $T(a, b, \lambda)$ are in involution with respect to either of these brackets, which ensures the complete (Liouville) integrability of both the flow TL and the map DTL. All three Poisson brackets have an r-matrix origin, i.e. they arise from different r-matrix Poisson brackets on the matrix algebras, the matrices $T(a, b, \lambda)$ forming a Poisson submanifold for all three brackets (cf [4, 5]).

We now give explicit formulae for the three Poisson brackets in the variables (a, b) (putting down only the non-vanishing brackets). The first ('linear') bracket reads:

$$\{a_k, b_k\}_1 = -\{a_k, b_{k+1}\}_1 = a_k. \tag{3.10}$$

The second ('quadratic') one:

$$\begin{aligned} &\{a_k, a_{k+1}\}_2 = -a_{k+1}a_k \qquad \{b_k, b_{k+1}\}_2 = -a_k \\ &\{a_k, b_k\}_2 = a_k b_k \qquad \{a_k, b_{k+1}\}_2 = -a_k b_{k+1}. \end{aligned} \tag{3.11}$$

The third ('cubic') one:

$$\begin{aligned} &\{a_k, a_{k+1}\}_3 = 2a_k a_{k+1} b_{k+1} \qquad \{b_k, b_{k+1}\}_3 = a_k(b_k + b_{k+1}) \\ &\{a_k, b_k\}_3 = -a_k(b_k^2 + a_k) \qquad \{a_k, b_{k+1}\}_3 = a_k(b_{k+1}^2 + a_k) \\ &\{a_k, b_{k+2}\}_3 = a_k a_{k+1} \qquad \{a_{k+1}, b_k\}_3 = -a_k a_{k+1}. \end{aligned} \tag{3.12}$$

The Hamiltonian functions generating the flow TL in these brackets are:

$$H^{(1)} = \tfrac{1}{2}\mathrm{tr}(T^2) \qquad H^{(2)} = \mathrm{tr}(T) \qquad H^{(3)} = \mathrm{tr}(\log(T)) = \log(\det(T)).$$

The first two of them have local expressions in terms of the variables (a, b), namely

$$H^{(1)} = \frac{1}{2}\sum_{k=1}^{N} b_k^2 + \sum_{k=1}^{N} a_k \qquad H^{(2)} = \sum_{k=1}^{N} b_k. \tag{3.13}$$

4. Parametrization of the linear bracket: remembering the Toda lattice case

The Toda lattice (1.2) admits a Lagrangian formulation with a Lagrange function

$$\mathcal{L}^{(1)}(x, \dot{x}) = \frac{1}{2}\sum_{k=1}^{N} \dot{x}_k^2 - \sum_{k=1}^{N} \exp(x_k - x_{k-1}). \tag{4.1}$$

A general procedure implies that the momenta p_k are given by

$$p_k = \partial\mathcal{L}^{(1)}/\partial\dot{x}_k = \dot{x}_k$$

so that the corresponding Hamiltonian function is

$$H^{(1)} = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \sum_{k=1}^{N} \exp(x_k - x_{k-1}) \tag{4.2}$$

and the system (1.2) takes the form of canonical equations of motion:

$$\begin{aligned} &\dot{x}_k = \partial H^{(1)}/\partial p_k = p_k \\ &\dot{p}_k = -\partial H^{(1)}/\partial x_k = \exp(x_{k+1} - x_k) - \exp(x_k - x_{k-1}). \end{aligned} \tag{4.3}$$

The connection with the flow TL is established by means of the following change of variables (Flaschka and Manakov):

$$a_k = \exp(x_{k+1} - x_k) \qquad b_k = p_k. \tag{4.4}$$

A simple calculation shows that the parametrization (4.4) leads immediately to the linear Poisson brackets (3.10) for the Flaschka variables (a, b). Moreover, under this parametrization the Hamiltonian function $H^{(1)}$ from (4.2) coincides with that given in (3.13).

The following well known statement is also from Flaschka and Manakov.

Proposition 1. If the Flaschka variables a_k, b_k are introduced according to the formulae (4.4), then their evolution induced by (4.3) coincides with the flow TL (3.1).

The proof of this proposition (as well as of the subsequent ones) consists of a straightforward check and will not be given in explicit detail.

So, the Newtonian equations of motion (1.2) admit a Lax representation (3.6) with the matrices (3.5) and (3.7), for the entries of which one has:

$$a_k = \exp(x_{k+1} - x_k) \qquad b_k = \dot{x}_k. \tag{4.5}$$

Let us now turn to the discrete-time case. Consider the equations of motion (1.5). It is easy to see that they admit a discrete Lagrangian formulation with the Lagrange function

$$\Lambda_1(\tilde{x}, x) = \sum_{k=1}^{N} \phi_1(\tilde{x}_k - x_k) - h \sum_{k=1}^{N} \exp(x_k - \tilde{x}_{k-1}) \tag{4.6}$$

where $\phi_1(\xi) = (\exp(\xi) - 1 - \xi)/h$. (This Lagrange function is a difference approximation to the continuous time one (4.1).) Hence the equations of motion (1.5) are equivalent to the symplectic map $(x, p) \mapsto (\tilde{x}, \tilde{p})$ with

$$hp_k = \exp(\tilde{x}_k - x_k) - 1 + h^2 \exp(x_k - \tilde{x}_{k-1}) \tag{4.7}$$

$$h\tilde{p}_k = \exp(\tilde{x}_k - x_k) - 1 + h^2 \exp(x_{k+1} - \tilde{x}_k). \tag{4.8}$$

We now demonstrate that they may be put in the form (3.2) (this result appeared first in [2]).

Proposition 2. If the variables a_k, b_k are defined by (4.4), then their discrete-time evolution induced by (4.7) and (4.8) coincides with the DTL (3.2), where the quantities β_k are defined as

$$\beta_k = \exp(\tilde{x}_k - x_k) \tag{4.9}$$

and satisfy the recurrent relation (3.3).

Note that the Poisson property of the map DTL, (3.2) and (3.3), with respect to the linear bracket (3.10) follows from this proposition.

5. Parametrization of the quadratic bracket: the case of the system (1.3)

We turn now to the system (1.3). First of all one sees that it admits a Lagrangian formulation with

$$\mathcal{L}^{(2)}(x, \dot{x}) = \sum_{k=1}^{N} \Psi(\dot{x}_k) - \sum_{k=1}^{N} \exp(x_k - x_{k-1}) \tag{5.1}$$

where

$$\Psi(\xi) = \xi \log(\xi) - \xi. \tag{5.2}$$

Hence the momenta p_k are introduced by

$$p_k = \partial \mathcal{L}^{(2)} / \partial \dot{x}_k = \log(\dot{x}_k)$$

the corresponding Hamiltonian function is equal to

$$H^{(2)} = \sum_{k=1}^{N} \exp(p_k) + \sum_{k=1}^{N} \exp(x_k - x_{k-1}) \tag{5.3}$$

and the canonical form of the equations of motion is:

$$\begin{aligned} \dot{x}_k &= \partial H^{(2)}/\partial p_k = \exp(p_k) \\ \dot{p}_k &= -\partial H^{(2)}/\partial x_k = \exp(x_{k+1} - x_k) - \exp(x_k - x_{k-1}). \end{aligned} \tag{5.4}$$

To establish this time the connection with the flow (3.1), one has to introduce the following change of variables:

$$a_k = \exp(x_{k+1} - x_k + p_k) \qquad b_k = \exp(p_k) + \exp(x_k - x_{k-1}). \tag{5.5}$$

It is easy to see that the parametrization (5.5) leads immediately to the quadratic Poisson brackets (3.11) for the Flaschka variables (a, b), and that the notation $H^{(2)}$ for the function (5.3) is consistent with (3.13).

Proposition 3. If the Flaschka variables a_k, b_k are introduced according to the formulae (5.5), then their evolution induced by (5.4) coincides with the flow TL (3.1).

So, the equations (1.3) admit a Lax representation (3.6) with the matrices (3.5) and (3.7), for the entries of which one has the formulae (5.5), which are also equivalent to

$$a_k = \dot{x}_k \exp(x_{k+1} - x_k) \qquad b_k = \dot{x}_k + \exp(x_k - x_{k-1}). \tag{5.6}$$

Turning to the discrete-time system (1.7), we find the following results. It admits a Lagrangian formulation with

$$\Lambda_2(\tilde{x}, x) = \sum_{k=1}^{N} \phi(\tilde{x}_k - x_k) - \sum_{k=1}^{N} \psi(x_k - \tilde{x}_{k-1}) \tag{5.7}$$

where the two functions $\phi(\xi)$, $\psi(\xi)$ are defined by

$$\phi(\xi) = \int_0^\xi \log\left|\frac{\exp(\eta) - 1}{h}\right| \mathrm{d}\eta \qquad \psi(\xi) = \int_0^\xi \log(1 + h\exp(\eta))\, \mathrm{d}\eta. \tag{5.8}$$

Hence a symplectic map $(x, p) \mapsto (\tilde{x}, \tilde{p})$ generated by (1.7) may be defined by the following relations:

$$h\exp(p_k) = (\exp(\tilde{x}_k - x_k) - 1)(1 + h\exp(x_k - \tilde{x}_{k-1})) \tag{5.9}$$

$$h\exp(\tilde{p}_k) = (\exp(\tilde{x}_k - x_k) - 1)(1 + h\exp(x_{k+1} - \tilde{x}_k)). \tag{5.10}$$

This map can be again reduced to (3.2) and (3.3)!

Proposition 4. If the variables a_k, b_k are defined by (5.5), then their evolution induced by (5.9) and (5.10) coincides with DTL (3.2), where the quantities β_k are given by

$$\beta_k = \exp(\tilde{x}_k - x_k)(1 + h\exp(x_k - \tilde{x}_{k-1})) \tag{5.11}$$

and satisfy the recurrent relation (3.3).

This proposition implies that the map (3.2) and (3.3) is Poisson with respect to the quadratic bracket (3.11).

6. Parametrization of the mixed bracket

There exists a nice parametrization of the Flaschka variables, resulting in a Poisson bracket which is a linear combination of the linear and the quadratic ones. We give in this section two applications of this parametrization for continuous time and discrete time.

Consider first the one-parametric family of lattices (belonging, of course, to the family (1.1)):

$$\ddot{x}_k = (1+\alpha\dot{x}_k)(\exp(x_{k+1}-x_k)-\exp(x_k-x_{k-1})). \tag{6.1}$$

The general procedure prescribes first to find the Lagrange function:

$$\mathcal{L}^{(\mathrm{m})}(x,\dot{x}) = \sum_{k=1}^{N}\alpha^{-2}\Psi(1+\alpha\dot{x}_k) - \sum_{k=1}^{N}\exp(x_k-x_{k-1}) \tag{6.2}$$

with the function Ψ given in (5.2). Then the momenta p_k are given by

$$p_k = \partial\mathcal{L}^{(\mathrm{m})}/\partial\dot{x}_k = \alpha^{-1}\log(1+\alpha\dot{x}_k)$$

the corresponding Hamiltonian function is equal to

$$H^{(\mathrm{m})} = \sum_{k=1}^{N}\alpha^{-2}\exp(\alpha p_k) - \sum_{k=1}^{N}\alpha^{-1}p_k + \sum_{k=1}^{N}\exp(x_k-x_{k-1}) \tag{6.3}$$

and the canonical form of the equations of motion is:

$$\begin{aligned}\dot{x}_k &= \partial H^{(\mathrm{m})}/\partial p_k = (\exp(\alpha p_k)-1)/\alpha\\ \dot{p}_k &= -\partial H^{(\mathrm{m})}/\partial x_k = \exp(x_{k+1}-x_k)-\exp(x_k-x_{k-1}).\end{aligned} \tag{6.4}$$

This time the connection with the flow TL is established by the following change of variables

$$a_k = \exp(x_{k+1}-x_k+\alpha p_k) \qquad b_k = \frac{\exp(\alpha p_k)-1}{\alpha} + \alpha\exp(x_k-x_{k-1}). \tag{6.5}$$

By a direct calculation one sees that this parametrization leads to the folllowing Poisson brackets for the Flaschka variables:

$$\begin{aligned}&\{b_{k+1},b_k\} = \alpha a_k \qquad \{a_{k+1},a_k\} = \alpha a_{k+1}a_k\\ &\{b_{k+1},a_k\} = a_k+\alpha b_{k+1}a_k \qquad \{b_k,a_k\} = -a_k-\alpha b_k a_k\end{aligned} \tag{6.6}$$

which is exactly a linear combination $\{\cdot,\cdot\}_1+\alpha\{\cdot,\cdot\}_2$.

Proposition 5. If the Flaschka variables a_k, b_k are introduced according to the formulae (6.5), then their evolution induced by (6.4) coincides with the flow TL (3.1).

A discretization of the lattice (6.1) reads

$$\frac{\alpha(\exp(\tilde{x}_k-x_k)-1)+h}{\alpha(\exp(x_k-\underset{\sim}{x}_k)-1)+h} = \frac{1+h\alpha\exp(\underset{\sim}{x}_{k+1}-x_k)}{1+h\alpha\exp(x_k-\tilde{x}_{k-1})}. \tag{6.7}$$

It is now not very surprising that this discrete-time lattice is just a new parametrization of the same map DTL as before!

To demonstrate this, as usual, we first represent (6.7) as an (implicit) symplectic map in the canonically conjugated coordinates

$$h\exp(\alpha p_k) = (\alpha(\exp(\tilde{x}_k-x_k)-1)+h)(1+h\alpha\exp(x_k-\tilde{x}_{k-1})) \tag{6.8}$$

$$h\exp(\alpha\tilde{p}_k) = (\alpha(\exp(\tilde{x}_k-x_k)-1)+h)(1+h\alpha\exp(x_{k+1}-\tilde{x}_k)). \tag{6.9}$$

Then the following statement holds:

Proposition 6. If the variables a_k, b_k are defined by (6.5), then their evolution induced by (6.8) and (6.9) coincides with DTL (3.2), where the quantities β_k are given by

$$\beta_k = \exp(\tilde{x}_k - x_k)(1 + h\alpha \exp(x_k - \tilde{x}_{k-1})) \tag{6.10}$$

and satisfy the recurrent relation (3.3).

An interesting particular case of the discrete-time lattice (6.7) arises, when

$$\alpha = h$$

so that the parameter α becomes small. Then (6.7) serves as a finite difference approximation to the lattice (6.1), which in turn is in this case an approximation to the usual Toda lattice (1.2). Hence we arrive at the discretization (1.6) of the Toda lattice which is different from (1.5). For completeness, we specialize the above formulae for this important particular case. A Lagrangian formulation of the system (1.6) is given by the Lagrange function

$$\Lambda_{\mathrm{m}}(\tilde{x}, x) = \sum_{k=1}^{N} \frac{1}{2h}(\tilde{x}_k - x_k)^2 - \sum_{k=1}^{N} \phi_2(x_k - \tilde{x}_{k-1}) \tag{6.11}$$

where $\phi_2(\xi) = h^{-1}\int_0^\xi \log(1 + h^2\exp(\eta))\,\mathrm{d}\eta$. This function serves as a finite-difference approximation to (4.1), different from (4.6). An equivalent form of writing (1.6) in canonically conjugated variables (x, p), following from the Lagrangian formulation, is:

$$\exp(hp_k) = \exp(\tilde{x}_k - x_k)(1 + h^2\exp(x_k - \tilde{x}_{k-1})) \tag{6.12}$$

$$\exp(h\tilde{p}_k) = \exp(\tilde{x}_k - x_k)(1 + h^2\exp(x_{k+1} - \tilde{x}_k)). \tag{6.13}$$

Proposition 7. If the variables a_k, b_k are defined by (6.5) with $\alpha = h$, i.e. by

$$a_k = \exp(x_{k+1} - x_k + hp_k) \qquad b_k = \frac{\exp(hp_k) - 1}{h} + h\exp(x_k - x_{k-1}) \tag{6.14}$$

then their evolution induced by (6.12) and (6.13) coincides with DTL (3.2), where the quantities β_k are given by

$$\beta_k = \exp(\tilde{x}_k - x_k)(1 + h^2\exp(x_k - \tilde{x}_{k-1})) = \exp(hp_k) \tag{6.15}$$

and satisfy the recurrent relation (3.3).

7. Parametrization of the cubic bracket: the case of the system (1.4)

Now we perform an analogous analysis for the system (1.4) and its discretization (1.8).

Starting with the continuous-time case, we first look for a Lagrangian formulation. The corresponding Lagrange function is readily seen to be equal to

$$\mathcal{L}^{(3)}(x, \dot{x}) = \sum_{k=1}^{N} \log(\dot{x}_k) - \sum_{k=1}^{N} \log(\sinh(x_k - x_{k-1})). \tag{7.1}$$

The momenta p_k are given by

$$p_k = \partial\mathcal{L}^{(3)}/\partial\dot{x}_k = 1/\dot{x}_k$$

the corresponding Hamiltonian function is equal to

$$H^{(3)} = \sum_{k=1}^{N} \log(p_k) + \sum_{k=1}^{N} \log(\sinh(x_k - x_{k-1})) \tag{7.2}$$

and the canonical form of the equations of motion is:

$$\dot{x}_k = \partial H^{(3)}/\partial p_k = 1/p_k$$
$$\dot{p}_k = -\partial H^{(3)}/\partial x_k = \coth(x_{k+1} - x_k) - \coth(x_k - x_{k-1}). \qquad (7.3)$$

The link with the flow (3.1) is established by means of the following change of variables:

$$a_k = \frac{1}{p_k p_{k+1} \sinh^2(x_{k+1} - x_k)}$$
$$b_k = -\frac{1}{p_k}[\coth(x_{k+1} - x_k) + \coth(x_k - x_{k-1})]$$
$$= -\frac{1}{p_k} \frac{\sinh(x_{k+1} - x_{k-1})}{\sinh(x_{k+1} - x_k)\sinh(x_k - x_{k-1})}. \qquad (7.4)$$

The parametrization (7.4) seems to appear for the first time in [6], where it was found that in this parametrization the flow of the Toda hierarchy with the Hamiltonian function $H = \mathrm{tr}(T^{-2})$ coincides with the so-called peakons lattice.

A direct calculation shows that the parametrization (7.4) results in the cubic Poisson brackets (3.12) for the Flaschka variables (a, b). It can also be demonstrated that the function (7.2) is exactly $H^{(3)} = \log(\det(T))$ expressed in terms of the canonically conjugated variables introduced by (7.4).

Proposition 8. If the Flaschka variables a_k, b_k are introduced according to (7.4), then their evolution induced by (7.3) coincides with the flow TL (3.1).

Hence, the lattice (1.4) admits a Lax representation (3.6) with the matrices (3.5) and (3.7), entries of which are given by (7.4), which are also equivalent to

$$a_k = \frac{\dot{x}_{k+1}\dot{x}_k}{\sinh^2(x_{k+1} - x_k)} \qquad b_k = -\frac{\dot{x}_k \sinh(x_{k+1} - x_{k-1})}{\sinh(x_{k+1} - x_k)\sinh(x_k - x_{k-1})}. \qquad (7.5)$$

For the discrete-time system (1.8) we find the following results. It admits a Lagrangian formulation with

$$\Lambda_3(\tilde{x}, x) = \sum_{k=1}^{N} \log(\sinh(\tilde{x}_k - x_k)) - \sum_{k=1}^{N} \log(\sinh(x_k - \tilde{x}_{k-1})). \qquad (7.6)$$

Hence a symplectic map $(x, p) \mapsto (\tilde{x}, \tilde{p})$ generated by (1.8) may be presented as a set of the following two relations:

$$p_k = h[\coth(\tilde{x}_k - x_k) + \coth(x_k - \tilde{x}_{k-1})] = \frac{h \sinh(\tilde{x}_k - \tilde{x}_{k-1})}{\sinh(\tilde{x}_k - x_k)\sinh(x_k - \tilde{x}_{k-1})} \qquad (7.7)$$

$$\tilde{p}_k = h[\coth(\tilde{x}_k - x_k) + \coth(x_{k+1} - \tilde{x}_k)] = \frac{h \sinh(x_{k+1} - x_k)}{\sinh(\tilde{x}_k - x_k)\sinh(x_{k+1} - \tilde{x}_k)}. \qquad (7.8)$$

And again this map can be reduced to (3.2) and (3.3)!

Proposition 9. If the variables a_k, b_k are defined by (7.4), then their evolution induced by (7.7) and (7.8) coincides with DTL (3.2), where the quantities β_k are given by

$$\beta_k = \frac{\sinh(x_{k+1} - \tilde{x}_k)\sinh(x_k - \tilde{x}_{k-1})}{\sinh(x_{k+1} - x_k)\sinh(\tilde{x}_k - \tilde{x}_{k-1})} \qquad (7.9)$$

and satisfy the recurrent relation (3.3).

This proposition implies that the map (3.2) and (3.3) is Poisson with respect to the cubic bracket (3.12).

8. Explicit discretizations

The structure of the equations (1.5), (1.6), (1.7) and (1.8) allows the following trick to be performed: rename $x_k(t)$ to $x_k(t-kh)$. Then $\underset{\sim}{x}_{k+1}$, $\tilde{x}_{k-1}$ on the right-hand sides will be replaced by x_{k+1}, x_{k-1}, and the following discrete-time lattice systems arise:

$$\exp(\tilde{x}_k - x_k) - \exp(x_k - \underset{\sim}{x}_k) = h^2(\exp(x_{k+1} - x_k) - \exp(x_k - x_{k-1})) \tag{8.1}$$

$$\exp(\tilde{x}_k - 2x_k + \underset{\sim}{x}_k) = \frac{1 + h^2\exp(x_{k+1} - x_k)}{1 + h^2\exp(x_k - x_{k-1})} \tag{8.2}$$

$$\frac{\exp(\tilde{x}_k - x_k) - 1}{\exp(x_k - \underset{\sim}{x}_k) - 1} = \frac{1 + h\exp(x_{k+1} - x_k)}{1 + h\exp(x_k - x_{k-1})} \tag{8.3}$$

and

$$\coth(\tilde{x}_k - x_k) - \coth(x_k - \underset{\sim}{x}_k) = \coth(x_{k+1} - x_k) - \coth(x_k - x_{k-1}). \tag{8.4}$$

Here (8.1) is the Hirota's discrete-time Toda lattice [7], (8.2) is a standard-like discretization of the Toda lattice introduced in [8], and the other two systems seem to be new.

These new discretizations are equivalent to those studied in the previous sections, when considered as equations on the lattice with the coordinates (t, k). However, the renaming of $x_k(t)$ to $x_k(t-kh)$ mixes the 'spatial' and 'temporal' variables, and this changes the properties of the *initial value problem*, which we are concerned with, dramatically.

First, from a practical point of view we must remark that the new models are explicit with respect to $\tilde{x}_k$, while the previous models require certain nonlinear algebraic equations to be solved (or, equivalently, continued fractions to be evaluated) in order to obtain the $\tilde{x}_k$.

Another important difference between our new models and the old ones lies in their algebraic, r-matrix structure. We have seen that the natural phase space for the old models is the orbit (set of the Lax matrices) of the usual Toda lattice. Now we intend to demonstrate that in the same sense the natural phase space for all four explicit discretizations is the orbit (the set of the Lax matrices) of the *relativistic* Toda lattice. For the system (8.2) this was first observed in [8], and for the system (8.1)—in [3]. Here we recall these results and prove the analogous statements for the systems (8.3) and (8.4).

More precisely, we will demonstrate that all four explicit discretizations are nothing more then four different appearances of the following system of difference equations, called hereafter DRTL:

$$\tilde{d}_k + h^2\tilde{c}_{k-1} = d_k + h^2 c_k \qquad \tilde{d}_{k+1}c_k = d_k\tilde{c}_k. \tag{8.5}$$

An equivalent form of DRTL may be obtained, if one resolves (8.5) for $(\tilde{c}_k, \tilde{d}_k)$:

$$\tilde{d}_k = d_{k-1}\frac{d_k + h^2 c_k}{d_{k-1} + h^2 c_{k-1}} \qquad \tilde{c}_k = c_k\frac{d_{k+1} + h^2 c_{k+1}}{d_k + h^2 c_k}. \tag{8.6}$$

The map defined by these difference equations is Poisson with respect to three different compatible Poisson brackets: a linear one

$$\{c_k, d_{k+1}\}_1 = -c_k \qquad \{c_k, d_k\}_1 = c_k \qquad \{d_k, d_{k+1}\}_1 = h^2 c_k \tag{8.7}$$

a quadratic one

$$\{c_k, c_{k+1}\}_2 = -c_k c_{k+1} \qquad \{c_k, d_{k+1}\}_2 = -c_k d_{k+1} \qquad \{c_k, d_k\}_2 = c_k d_k \tag{8.8}$$

and a cubic one

$$\{c_k, c_{k+1}\}_3 = c_k c_{k+1}(2d_{k+1} + h^2 c_k + h^2 c_{k+1}) \qquad \{d_k, d_{k+1}\}_3 = h^2 c_k d_k d_{k+1}$$
$$\{c_k, d_k\}_3 = -c_k d_k(d_k + h^2 c_k) \qquad \{c_k, d_{k+1}\}_3 = c_k d_{k+1}(d_{k+1} + h^2 c_k)$$

$$\{c_k, d_{k+2}\}_3 = h^2 c_k c_{k+1} d_{k+2} \qquad \{c_{k+1}, d_k\}_3 = -h^2 c_k c_{k+1} d_k$$
$$\{c_k, c_{k+2}\}_3 = h^2 c_k c_{k+1} c_{k+2}. \tag{8.9}$$

The Lax representation for the map (8.5) may be given in terms of the $N \times N$ matrices depending on the dynamical variables (c, d) and an additional parameter λ:

$$L(c, d, \lambda) = \sum_{k=1}^{N} d_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k} \tag{8.10}$$

$$U(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} - h\lambda^{-1} \sum_{k=1}^{N} c_k E_{k,k+1}. \tag{8.11}$$

It is easy to check that the difference equations (8.5) are equivalent to the matrix equation

$$U\tilde{L} = L\tilde{U} \qquad \text{or} \qquad \tilde{L}\tilde{U}^{-1} = U^{-1}L. \tag{8.12}$$

In terms of the Lax matrix

$$T(c, d, \lambda) = L(c, d, \lambda)U^{-1}(c, d, \lambda) \tag{8.13}$$

(8.12) takes the form

$$\tilde{T} = U^{-1}TU = L^{-1}TL \tag{8.14}$$

which implies, in particular, that the spectral invariants of the matrix T are integrals of motion for the map (8.5).

As observed in [5, 8], the matrix T from (8.13) serves as the Lax matrix of the *relativistic Toda hierarchy* (which is also tri-Hamiltonian with respect to the brackets (8.7), (8.8) and (8.9)).

Now we recall how (8.1) and (8.2) can be reduced to DRTL (8.5), and then show that the same is true for (8.3) and (8.4).

We start with (8.1). It is easy to find a Lagrangian formulation of these equations with a Lagrange function

$$\Lambda_4(\tilde{x}, x) = \sum_{k=1}^{N} \phi_1(\tilde{x}_k - x_k) - h \sum_{k=1}^{N} \exp(\tilde{x}_k - \tilde{x}_{k-1}) \tag{8.15}$$

(where, as in section 4, $\phi_1(\xi) = (\exp(\xi) - 1 - \xi)/h$). Hence (8.1) is equivalent to a symplectic map $(x, p) \mapsto (\tilde{x}, \tilde{p})$ with

$$hp_k = \exp(\tilde{x}_k - x_k) - 1 \tag{8.16}$$
$$h\tilde{p}_k = \exp(\tilde{x}_k - x_k) - 1 + h^2 \exp(\tilde{x}_{k+1} - \tilde{x}_k) - h^2 \exp(\tilde{x}_k - \tilde{x}_{k-1}). \tag{8.17}$$

Proposition 10. Let the coordinates (c, d) be parametrized by the canonically conjugated variables (x, p) according to the formulae

$$c_k = \exp(x_{k+1} - x_k) \qquad d_k = 1 + hp_k - h^2 \exp(x_{k+1} - x_k). \tag{8.18}$$

Then their discrete-time evolution induced by (8.16) and (8.17) coincides with the DRTL (8.5).

It is important to notice that the parametrization (8.18) results (up to the factor h) in the linear Poisson bracket (8.7), which proves independently that the map DRTL (8.6) is Poisson with respect to this bracket.

Turning now to (8.2), we find a Lagrangian formulation of these equations with

$$\Lambda_5(\tilde{x}, x) = \sum_{k=1}^{N} \frac{1}{2h}(\tilde{x}_k - x_k)^2 - \sum_{k=1}^{N} \phi_2(x_k - x_{k-1}) \tag{8.19}$$

where, as in the section 6,

$$\phi_2(\xi) = h^{-1}\int_0^{\xi} \log(1 + h^2 \exp(\eta))\, d\eta.$$

Hence the expression for the momenta p_k and their updates, equivalent to (8.2), are:

$$\exp(hp_k) = \exp(\tilde{x}_k - x_k)\frac{1 + h^2 \exp(x_k - x_{k-1})}{1 + h^2 \exp(x_{k+1} - x_k)} \tag{8.20}$$

$$\exp(h\tilde{p}_k) = \exp(\tilde{x}_k - x_k). \tag{8.21}$$

Proposition 11. Let the coordinates (c, d) be parametrized by the canonically conjugated varibles (x, p) according to the formulae

$$c_k = \exp(x_{k+1} - x_k + hp_k) \qquad d_k = \exp(hp_k). \tag{8.22}$$

Then their discrete-time evolution induced by (8.20) and (8.21) coincides with DRTL (8.5).

Notice that (8.22) results (up to the factor h) in the quadratic Poisson bracket (8.8), which proves independently that the map DRTL (8.6) is Poisson with respect to this bracket.

It remains to perform analogous considerations for explicit discretizations (8.3) and (8.4) of the lattices (1.3) and (1.4). As already pointed out, these systems turn out to be further realizations of the same map DRTL (8.6)!

As for the system (8.3), it is easy to find a Lagrangian formulation for it with

$$\Lambda_6(\tilde{x}, x) = \sum_{k=1}^{N} \phi(\tilde{x}_k - x_k) - \sum_{k=1}^{N} \psi(x_k - x_{k-1}) \tag{8.23}$$

where $\phi(\xi)$ and $\psi(\xi)$ are defined by 5.8). Hence a Hamiltonian formulation of this system is given by:

$$h\exp(p_k) = (\exp(\tilde{x}_k - x_k) - 1)\frac{1 + h\exp(x_k - x_{k-1})}{1 + h\exp(x_{k+1} - x_k)} \tag{8.24}$$

$$h\exp(\tilde{p}_k) = (\exp(\tilde{x}_k - x_k) - 1). \tag{8.25}$$

Proposition 12. Let the coordinates (c, d) be parametrized by the canonically conjugated varibles (x, p) according to the formulae

$$c_k = \exp(x_{k+1} - x_k + p_k) \qquad d_k = 1 + h\exp(p_k) + h\exp(x_k - x_{k-1}). \tag{8.26}$$

Then their discrete-time evolution induced by (8.24) and (8.25) coincides with DRTL (8.5).

It is easy to calculate that the parametrization (8.26) generates the following Poisson bracket:

$$\begin{aligned} &\{c_{k+1}, c_k\} = c_{k+1}c_k \qquad \{d_{k+1}, d_k\} = h^2 c_k \\ &\{d_k, c_k\} = -c_k(d_k - 1) \qquad \{d_{k+1}, c_k\} = c_k(d_{k+1} - 1). \end{aligned} \tag{8.27}$$

This is, obviously, $\{\cdot,\cdot\}_2 - \{\cdot,\cdot\}_1$, a linear combination of the brackets (8.7) and (8.8). Of course, the Poisson property of the map DRTL with respect to this bracket follows from the previous results, but proposition 12 gives an alternative way to prove this.

Finally we discuss the system (8.4). It has a Lagrangian representation with a Lagrange function

$$\Lambda_7(\tilde{x}, x) = \sum_{k=1}^{N} \log(\sinh(\tilde{x}_k - x_k)) - \sum_{k=1}^{N} \log(\sinh(\tilde{x}_k - \tilde{x}_{k-1})). \tag{8.28}$$

Hence a symplectic map $(x, p) \mapsto (\tilde{x}, \tilde{p})$ generated by (8.4) is equivalent to a set of the following two relations:

$$p_k = h \coth(\tilde{x}_k - x_k) \tag{8.29}$$

$$\tilde{p}_k = h[\coth(\tilde{x}_k - x_k) + \coth(\tilde{x}_{k+1} - \tilde{x}_k) - \coth(\tilde{x}_k - \tilde{x}_{k-1})]. \tag{8.30}$$

Proposition 13. Let the coordinates (c, d) be parametrized by the canonically conjugated variables (x, p) according to the formulae

$$c_k = \frac{1}{(p_k + h\coth(x_k - x_{k-1}))(p_{k+1} + h\coth(x_{k+1} - x_k))\sinh^2(x_{k+1} - x_k)}$$
$$d_k = \frac{p_k - h\coth(x_{k+1} - x_k)}{p_k + h\coth(x_k - x_{k-1})}. \tag{8.31}$$

Then their discrete-time evolution induced by (8.29) and (8.30) coincides with DRTL (8.5).

A direct, however somewhat tedious calculation, shows that the parametrization (8.31) leads to the following Poisson brackets:

$$\{c_k, c_{k+1}\} = 2h^{-1}c_kc_{k+1}(d_{k+1} - 1) + hc_kc_{k+1}(c_k + c_{k+1})$$
$$\{d_k, d_{k+1}\} = hc_k(d_kd_{k+1} - 1)$$
$$\{c_k, d_k\} = -h^{-1}c_k(d_k - 1)^2 - hc_k^2d_k \qquad \{c_k, d_{k+1}\} = h^{-1}c_k(d_{k+1} - 1)^2 + hc_k^2d_{k+1}$$
$$\{c_k, d_{k+2}\} = hc_kc_{k+1}d_{k+2} \qquad \{c_{k+1}, d_k\} = -hc_kc_{k+1}d_k \qquad \{c_k, c_{k+2}\}_3 = hc_kc_{k+1}c_{k+2}.$$

This is $h^{-1}(\{\cdot,\cdot\}_3 + 2\{\cdot,\cdot\}_2 - \{\cdot,\cdot\}_1)$, a linear combination of the brackets (8.7), (8.8) and (8.9). The Poisson property of the map DRTL with respect to this bracket follows from proposition 13. This implies the Poisson property with respect to (8.9), if one takes into account the previous results.

9. Conclusion

We have considered in this paper two recently introduced integrable lattices (1.3) and (1.4). We have demonstrated that they may be considered as Bäcklund transformations of the usual Toda lattice (1.2). These transformations consist of identifying the variables (a, b) in (4.4), in (4.5) and in (7.4), which may be viewed as transformations between three sets of variables (x, p) (and, consequently, between three sets of variables $(x, \dot{x})$).

For each of these systems one has different integrable discretizations. Some of them share the Lax matrix with the continuous-time prototype. These discretizations generate Newtonian equations implicit with respect to the updates $\tilde{x}_k$. Other discretizations have Lax representations with the Lax matrix defining the *relativistic* Toda hierarchy. These discretizations turn out to be explicit.

All implicit discretizations turn out to be connected by Bäcklund transformations. An underlying fact is that all of them appear from one and the same integrable map, if the relevant variables (a, b) are parametrized by canonically conjugated ones (x, p) in different ways, generating different Poisson brackets on the set of (a, b) (and hence on the set of Lax matrices).

Exactly the same holds true for the explicit discretizations.

We would like to note here that all the Poisson brackets on the sets of Lax matrices of the Toda and the relativistic Toda hierarchies were given an r-matrix interpretation in [4, 5].

Interesting open problems are suggested by the form of the Hamiltonian function (5.3), more specifically, by the form of its 'kinetic part' $\sum_{k=1}^{N}\exp(p_k)$. First, a natural question arises, whether there exist other integrable systems with such a kinetic term in the Hamiltonian, for example, systems analogous to the Calogero–Moser ones. Second, a quantization of such Hamiltonians will lead to integrable difference operators, which might be connected with interesting classes of special functions.

As a further interesting (and difficult) open problem we would like to mention the task of finding and classification of all integrable discrete-time Lagrangian systems (several examples of which are discussed in the present paper).

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